Characteristic classes of bad orbifold vector bundles[☆]

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Abstract

We show that every bad orbifold vector bundle can be realized as the restriction of a good orbifold vector bundle to a suborbifold of the base space. We give an explicit construction of this result in which the Chen–Ruan orbifold cohomologies of the two base spaces are isomorphic (as additive groups). This construction is used to indicate an extension of the Chern–Weil construction of characteristic classes to bad orbifold vector bundles. In particular, we apply this construction to the orbifold Euler class and demonstrate that it acts as an obstruction to the existence of nonvanishing sections.
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1. Introduction

When orbifolds were originally introduced by Satake (under the name V-manifold; see [3,4]), the definition given coincides with the modern definition of a reduced (codimension 2) orbifold. Recall that an orbifold Q is a Hausdorff topological space locally modeled on $\mathbb{R}^n/G$ where G is a finite group of automorphisms; Q is **reduced** if each local group G acts effectively and **unreduced** otherwise. Since each point of an unreduced orbifold is a fixed point for a nontrivial group element, the orbifold is composed entirely of singular points. Such orbifolds appear, for instance, as suborbifolds of reduced orbifolds consisting entirely of singular points.

Many of the techniques used to study the differential geometry of vector bundles over smooth manifolds extend readily to the case of orbifold vector bundles over smooth orbifolds provided that the orbifolds in question are reduced (see [2, Section 4.3]). In this case, every orbifold vector bundle is a good orbifold vector bundle. An orbifold vector bundle E over the orbifold Q is **good** if it is covered by charts of the form $\{V \times \mathbb{R}^k, G, \tilde{\pi}\}$ such that $\{V, \mathbb{R}^k, \pi\}$ is an orbifold chart over Q, and the kernel of the G-action on $V \times \mathbb{R}^k$ coincides with the kernel of the G-action on V. If E is a good vector bundle, then its geometry is identical to that of a vector bundle over a reduced orbifold. In fact, as

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noted by Chen and Ruan [2], the associated reduced orbifold E_{red} is an orbifold vector bundle over Q_{red}. Therefore, the fibers of E over an open dense subset of Q are vector spaces.

In the case that E is not good, it is called a **bad** orbifold vector bundle. Bad orbifold vector bundles have fibers given by $\mathbb{R}^n/G$ for a nontrivial action of a nontrivial group G over every point of the orbifold. Hence, as the sections of an orbifold vector bundle are required to take values in the fixed-point subspace of the fiber, the rank of the possible values of a section of a bad bundle is smaller than the rank of the bundle at every point. In some cases, such bundles admit only the zero section (see Example 13 below). Additionally, since bad orbifold vector bundles cannot be studied as vector bundles over reduced orbifolds, many of the natural maps defined between orbifolds cannot be used to pull back a bad orbifold vector bundle in a well-defined way.

In this paper, we suggest a method for extending the geometric techniques used to study good orbifold vector bundles to the case of bad orbifold vector bundles. Our main result is the following theorem.

Theorem 1. *For every bad orbifold vector bundle $\rho : E \to Q$, there is an orbifold R of which Q is a suborbifold and a good orbifold vector bundle $\hat{\rho} : \hat{E} \to R$ such that E is isomorphic to the restriction $\hat{E}_{|Q}$ of $\hat{E}$ to Q.*

As an application of this result, we demonstrate how the Chern–Weil description of characteristic classes can be used to define characteristic classes for bad orbifold vector bundles. Our primary focus is the Euler class, which we show plays the role of an obstruction to the existence of nonvanishing sections.

In Section 2, we develop the tools necessary to prove Theorem 1 and demonstrate an explicit construction of a good orbifold vector bundle that restricts to a given bad bundle E. Additionally, we show how this construction allows the Chern–Weil construction of characteristic classes. In Section 3, we apply these results to the Euler class and extend the obstruction property of the Euler class to the case of bad orbifold vector bundles. The reader is referred to [3,4,2,1] and [5] for relevant background material. In particular, [2] contains as an appendix a modern introduction to orbifolds; for the most part, we follow its notation.

2. Bad bundles as restrictions of good bundles

Let $\rho : E \to Q$ be a rank k orbifold vector bundle over the orbifold Q, which we do not assume to be reduced. Let K_b denote the (isomorphism class of the) finite group that acts trivially in each orbifold chart for Q, and let K_f denote the (isomorphism class of the) subgroup of K_b that acts trivially in orbifold vector bundle charts for E. Then E is bad precisely when K_f is a proper subgroup of K_b.

It is well known in the case of manifolds that the pullback via the projection of E over its own total space, $\rho^* E$, is isomorphic to the vertical tangent space VE. For bad orbifold bundles, however, the pullback $\rho^* E$ is not well defined. In fact, as $\Sigma_Q = Q$, the preimage of the regular points $\rho^{-1}(Q_{\text{reg}})$ is empty. Therefore, the projection is never a regular map, and the map ρ need not admit a unique compatible system (see [2, Section 4.4] for more details). However, in the case of a bad orbifold vector bundle, we do have that the vertical tangent bundle of E coincides with E when restricted to the zero section.

To make this statement precise, we begin with the following definition, similar to the manifold case.

Definition 2. Let $\rho : E \to Q$ be a rank k orbifold vector bundle over the orbifold Q. Let $\rho_T : TE \to E$ denote the orbifold tangent bundle of the orbifold E. A vector $X \in TE$ is **vertical** if, for each lift $\tilde{X}$ of X into the tangent bundle of $V \times \mathbb{R}^k$ for a chart $\{V \times \mathbb{R}^k, G, \tilde{\pi}\}$ for E, $\tilde{X}$ is a vertical vector for the trivial vector bundle $V \times \mathbb{R}^k$. In other words, $\tilde{X}(\rho^* f) = 0$ for each G-invariant $f \in \mathcal{C}^\infty(V)$ (note that we use ρ to denote the projection $\rho : V \times \mathbb{R}^k \to V$ in a chart as well as the projection for E). The collection of vertical vectors is the **vertical tangent bundle** $\rho_V : VE \to E$, where ρ_V is the restriction of ρ_T to VE.

It is clear that the definition of a vertical vector does not depend on the choice of chart, and that the set of vertical vectors forms a suborbifold vector bundle of TE.

Proposition 3. *Let $\rho : E \to Q$ be a vector bundle over the orbifold Q as above. Then if $\iota : Q \to E$ denotes the embedding of Q into E as the zero section, the restriction $VE_{|\iota(Q)}$ of VE to $\iota(Q)$ is naturally isomorphic to the bundle E.*

Proof. Choose a compatible cover of Q, and then for each injection $i : \{V_1, G_1, \pi_1\} \to \{V_2, G_2, \pi_2\}$, there is an associated transition map $g_i : V_1 \to \text{AUT}(\mathbb{R}^k)$ (see [2, Section 4.1]). Associated to each chart $\{V, G, \pi\}$ in the cover is a chart $\{V \times \mathbb{R}^k, G, \tilde{\pi}\}$ for E, and the resulting cover is a compatible cover of E. Moreover, each injection of charts for E is induced by an injection in Q by associating $i : \{V_1, G_1, \pi_1\} \to \{V_2, G_2, \pi_2\}$, consisting of an embedding $\phi : V_1 \to V_2$ and an injective homomorphism $\lambda : G_1 \to G_2$, with the injection $i_E : \{V_1 \times \mathbb{R}^k, G_1, \tilde{\pi}_1\} \to \{V_2 \times \mathbb{R}^k, G_2, \tilde{\pi}_2\}$, consisting of the embedding

$$\phi_E : V_1 \times \mathbb{R}^k \to V_2 \times \mathbb{R}^k$$
$$: (x, v) \mapsto (\phi(x), g_i(x)v)$$

and homomorphism $\lambda_E := \lambda$.

Using this compatible cover, the transition maps for TE are given by the differentials of the embeddings ϕ_E, so that for $(x, v) \in V_1 \times \mathbb{R}^k$, with respect to an injection i_E, we have $(g_i)_{TE}(x, v) = (d\phi_x, d(g_i(x))) \in \text{AUT}(\mathbb{R}^k) \times \text{AUT}(\mathbb{R}^k) \subseteq \text{AUT}(\mathbb{R}^{2k})$.

The vertical vectors of TE clearly correspond to elements of $V \times T\mathbb{R}^k \subseteq TV \times T\mathbb{R}^k = T(V \times \mathbb{R}^k)$. For the bundle VE, then, the transition maps are $(g_i)_{VE}(x, v) = d(g_i(x)) \in \text{AUT}(\mathbb{R}^k)$. Restricted to the zero section, we have $(g_i)_{VE_{|\iota(Q)}}(x, \mathbf{0}) = d(g_i(x))$, and as each $g_i(x)$ is linear, $d(g_i(x)) = g_i(x)$. Therefore, identifying $(x, \mathbf{0}) \in V_1 \times \mathbb{R}^k$ with $x \in V_1$, we see that the transition maps $(g_i)_{VE_{|\iota(Q)}}$ for $VE_{|\iota(Q)}$ are identical to the transition maps g_i for E. Therefore, the bundles are isomorphic. $\square$

With this, we realize the orbifold vector bundle E as a restriction of another orbifold vector bundle, VE, to a subset of the base. In fact, we have the following two lemmas.

Lemma 4. *The bundle $\rho_V : VE \to E$ is a good orbifold vector bundle.*

Proof. We have that VE is a subbundle of the tangent bundle TE of E. In each chart, the group action for the fibers of the tangent bundle is defined to be the differential of the group action on the base. Hence, a group element acts trivially on the fibers of TE if and only if it acts trivially on the base, and TE is good.

As the kernel of the group action on the fibers of an orbifold vector bundle is clearly contained in the kernel of the group action on the base space, any subbundle of a good orbifold vector bundle is clearly good. Therefore, as VE is a subbundle of TE, VE is a good orbifold vector bundle. $\square$

The reader is warned that E need not be a reduced orbifold.

Lemma 5. *The zero section $\iota(Q)$ is a suborbifold of E that is diffeomorphic to Q.*

Proof. The orbifold structure on E is defined by a set of orbifold charts $\{V \times \mathbb{R}^k, G, \tilde{\pi}\}$ for E induced by orbifold charts $\{V, G, \pi\}$ for Q. Each such chart restricts to a chart $\{V \times \{\mathbf{0}\}, G, \tilde{\pi}_{|V \times \{\mathbf{0}\}}\}$ for $\iota(Q)$. It is clear that $\{V \times \{\mathbf{0}\}, G, \tilde{\pi}_{|V \times \{\mathbf{0}\}}\}$ is isomorphic to $\{V, G, \pi\}$ as orbifold charts and that injections of charts for Q correspond bijectively to injections of charts for $\iota(Q)$.

In orbifold charts as above, ι has a well-defined $\mathcal{C}^\infty$ lifting $\tilde{\iota} : V \to V \times \{\mathbf{0}\}$ as the identity on V, taking the group homomorphisms to be the identity. When restricted to the image $\iota(Q)$, it is clear that this map is bijective and that its inverse admits a well-defined $\mathcal{C}^\infty$ lifting. $\square$

With this, we have proven Theorem 1. In the explicit construction given by Proposition 3, $R = E$ and $\hat{E} = VE$.

Recall that if U_p is uniformized by a **chart at** p (i.e. a chart $\{V_p, G_p, \pi_p\}$ such that G_p acts as a subgroup of $O(n)$ and p is the image of the origin in V_p) and if $q \in U_p$, then the conjugacy class $(g)_{G_q}$ of $g \in G_q$ is said to be equivalent to the conjugacy class $(h)_{G_p}$ of $h \in G_p$ if there is an injection i from a chart at q into the chart at p whose homomorphism λ maps g to h (see [1, Section 3.1]). Clearly, as the charts and injections for E can be taken to be those induced by its structure as a vector bundle over Q, we see that the set of equivalence classes for group elements of Q coincide with those for E. Hence, we let T denote the set of equivalence classes for both Q and E.

The embedding $\iota : Q \to E$ induces a natural $\mathcal{C}^\infty$ map on the spaces of sectors of these orbifolds (see [1, Definition 3.1.2]). In particular, for each sector $\tilde{E}_{(g)}$, let $\iota_{(g)} : \tilde{Q}_{(g)} \to \tilde{E}_{(g)}$ be defined by setting $\iota_{(g)}[(p, (g))] = [(p, \mathbf{0}), (g)]$. This map has a well-defined $\mathcal{C}^\infty$ lifting to the embedding of V^g into $(V \times \mathbb{R}^k)^g$ as the zero section in an orbifold chart. Hence, on each twisted sector $\tilde{E}_{(g)}$, it induces a restriction homomorphism

$$\iota^*_{(g)} : H^*_{dR}(\tilde{E}_{(g)}) \to H^*_{dR}(\tilde{Q}_{(g)})$$

in the de Rham cohomology of the sector. In the case that Q and E admit almost complex structures so that $H^*_{\mathrm{orb}}(Q)$ and $H^*_{\mathrm{orb}}(E)$ are defined, we let

$$
\begin{aligned}
\iota^* &:= \bigoplus_{(g)\in T} \iota^*_{(g)} \\
&: \quad H^*_{\mathrm{orb}}(E) \to H^*_{\mathrm{orb}}(Q)
\end{aligned}
$$

be the sum of these homomorphisms defined in Chen–Ruan orbifold cohomology (note that throughout, we use cohomology with real coefficients).

Lemma 6. *Suppose E and Q admit almost complex structures. Then the restriction homomorphism $\iota^* : H^*_{\mathrm{orb}}(E) \to H^*_{\mathrm{orb}}(Q)$ is an isomorphism of ungraded additive groups.*

The reader is warned that this isomorphism does not generally preserve the grading; the degree shifting number of a group element $g \in K_b$ is zero with respect to the orbifold structure on Q, while it need not be zero with respect to the orbifold structure of E (see Example 13 below).

Proof. A compatible cover of orbifold charts for $\tilde{E}$ can be taken as follows. Each chart for Q at a point p of the form $\{V_p, G_p, \pi_p\}$ again induces a chart $\{V_p \times \mathbb{R}^k, G_p, \tilde{\pi}_p\}$ for E. Then, for each equivalence class $(g) \in T$ with representative $g \in G_p$, there is a chart $\{(V_p \times \mathbb{R}^k)^g, C(g), \tilde{\pi}_{p,g}\}$ for a neighborhood of $[(p, \mathbf{v}), (g)] \in \tilde{E}_{(g)}$.

Now, consider the map

$$
\begin{aligned}
H_t &: \tilde{E}_{(g)} \to \tilde{E}_{(g)} \\
&: [(p, \mathbf{v}), (g)] \mapsto [(p, t\mathbf{v}), (g)]
\end{aligned}
$$

for $t \in [0, 1]$. It is clear that this map defines a $\mathcal{C}^\infty$ deform retraction of $\tilde{E}_{(g)}$ onto the image of $\iota_{(g)}$. Therefore, $\iota^*_{(g)}$ is an isomorphism between $H^*_{dR}(\tilde{E}_{(g)})$ and $H^*_{dR}(\tilde{Q}_{(g)})$. As $H^*_{\mathrm{orb}}(Q)$ and $H^*_{\mathrm{orb}}(E)$ are defined to be sums of the de Rham cohomology of the sectors of the respective orbifolds, the sum, ι^*, is then clearly an isomorphism of additive groups. $\square$

Finally, we note that any characteristic class c (see [2, Proposition 4.3.4]) or orbifold characteristic class c_{orb} (see [5, Appendix B]) defined by the Chern–Weil construction for vector bundles can be extended to the case of bad orbifold vector bundles. Note that the definition of orbifold characteristic classes $c_{\mathrm{orb}}(E)$ applies the Chern–Weil construction to connections on the bundle $\tilde{E}$ over $\tilde{Q}$.

Definition 7. If c is a characteristic class, we define

$$
c(E) := \iota^* c(VE)
$$

to take its value in $H^*_{dR}(Q)$. In the case that E and Q admit almost complex structures, we define

$$
c_{\mathrm{orb}}(E) := \iota^* c_{\mathrm{orb}}(VE) = \iota^* c(\widetilde{VE})
$$

to take its value in $H^*_{\mathrm{orb}}(Q)$.

As these definitions are given in terms of good bundles, they satisfy the expected properties. The following two lemmas show that they coincide with the original definitions when those are defined.

Lemma 8. *Let E be a good orbifold vector bundle and let $c(E)$ denote a characteristic defined by the Chern–Weil construction. Then Definition 7 coincides with the previous definition of $c(E)$: $\iota^* c(VE) = c(E)$.*

Proof. As E is good, E_{red} is naturally an orbifold vector bundle over Q_{red} with identical geometry to that of E (see [2]). Hence, without loss of generality, we assume Q is reduced.

The projection ρ is a regular map, so that it admits a unique isomorphism class of compatible systems (see [2, Lemma 4.4.11 and Remark 4.4.12b]). Hence, the pullback bundle $\rho^* E$ of E is well defined.

Note that in this compatible system, a chart $\{V \times \mathbb{R}^k, G, \tilde{\pi}\}$ for $E_{|U}$ is associated to the chart $\{V, G, \pi\}$ for $U \subseteq Q$ from which it was induced. The lift $\tilde{\rho}_{U, E_{|U}} : V \times \mathbb{R}^k \to V$ of ρ is simply the projection onto the first factor, and the injection $\lambda(i)$ associated to an injection

$$
i : \{V_1 \times \mathbb{R}^k, G_1, \tilde{\pi}_1\} \to \{V_2 \times \mathbb{R}^k, G_2, \tilde{\pi}_2\}
$$

is the obvious injection

$$\lambda(i) : \{V_1, G_1, \pi_1\} \rightarrow \{V_2, G_2, \pi_2\}$$

where the embedding is restricted to the zero section and the group homomorphism of $\lambda(i)$ is taken to be that of i.

An argument similar to that used in Proposition 3 shows that in this case, ρ^*E is isomorphic to VE as orbifold bundles over E. If $g_{\lambda(i)}$ denotes the transition map for E corresponding to an injection $\lambda(i)$ (which can be taken to be associated to an injection i for E), then the corresponding transition map on the vertical tangent bundle is again given by $(g_{\lambda(i)})_{VE}(x, v) = d(g_{\lambda(i)}(x)) \in \text{AUT}(\mathbb{R}^k)$. The transition maps of ρ^*E are the pullbacks of the transition maps via the lifts of ρ given above:

$$(g_i)_{\rho^*E} = g_{\lambda(i)} \circ \tilde{\rho}_{U, E_{|U}}.$$

Hence, the transition maps for ρ^*E are constant along the fibers of E and are seen to be equal to the transition maps for VE.

With this, as

$$\rho^*c(E) = c(\rho^*E) \tag{1}$$

by [2, Lemma 4.4.3], we have

$$\begin{aligned}
\iota^*c(VE) &= \iota^*c(\rho^*E) \\
&= \iota^*\rho^*c(E) \quad \text{(by Eq. (1))} \\
&= (\rho \circ \iota)^*c(E) \\
&= c(E),
\end{aligned}$$

as $\rho \circ \iota$ is the identity on Q up to the diffeomorphism given by Lemma 5. $\quad\square$

Lemma 9. *Definition 7 extends the original definition of orbifold characteristic classes: if c_{orb} is defined as in [5, Lemma 4.4.1] for a vector bundle E, then $\iota^*c_{\text{orb}}(VE) = c_{\text{orb}}(E)$.*

Proof. In this case, for each $(g) \in T$, the bundle $\rho_{(g)} : \tilde{E}_{(g)} \rightarrow \tilde{Q}_{(g)}$ is a good bundle. Hence, we can apply Lemma 8 to see that (the reduction of) $\tilde{E}_{(g)}$ pulls back via its projection to a bundle isomorphic to its vertical tangent bundle. Moreover, we have that $c(\tilde{E}_{(g)}) = \iota^*c(V\tilde{E}_{(g)})$.

With this, we need only note that $c_{\text{orb}}(E)$ is the sum $\sum_{(g) \in T} c(\tilde{E}_{(g)})$, so that equality on each sector implies that $\iota^*c_{\text{orb}}(VE) = c_{\text{orb}}(E)$. $\quad\square$

3. Application: The orbifold Euler class of a bad orbifold vector bundle

In previous work (see [5,6]), we have defined an **orbifold Euler class** in Chen–Ruan orbifold cohomology (note that in [5], the definition of an orbifold vector bundle was taken to be that of a good orbifold vector bundle). In [6], it was demonstrated that when all of the local groups are cyclic, this class acts as a complete obstruction to the existence of nonvanishing tangent vector fields.

In this section, we show how the results of the last section can be used to extend some of the techniques used to study good orbifold vector bundles to the case of bad orbifold vector bundles. In particular, we extend the definition of $e_{\text{orb}}(E)$ to the case of E bad and show that it acts as an obstruction to the existence of nonvanishing sections. We begin with the following lemma.

Lemma 10. *The bundle $\rho : E \rightarrow Q$ admits a nonvanishing section if and only if $\rho_{VE} : VE \rightarrow E$ admits a nonvanishing section.*

Proof. Suppose VE admits a nonvanishing section s, and then the restriction of s to the zero section $\iota(Q)$ of E is a nonvanishing section of $VE_{|\iota(Q)}$. Using the bundle isomorphism between $VE_{|\iota(Q)}$ and E given in Proposition 3, $s_{|\iota(Q)}$ defines a nonvanishing section of E.

Conversely, suppose E admits a nonvanishing section s. Define $\hat{s}$ on $\iota(Q) \subset E$ using the isomorphism between E and $VE_{|\iota Q}$ and extend $\hat{s}$ to E by defining its value to be constant along each fiber of E within each chart of the form

$\{V \times \mathbb{R}^k, G, \tilde{\pi}\}$. The result is clearly G-invariant, and hence defines a vertical vector field $\hat{s}$ on E that is smooth and nonvanishing. $\quad \square$

With this, we extend the definition of the orbifold Euler class to the case of bad orbifold vector bundles.

Definition 11. Let $\rho : E \to Q$ be an orbifold vector bundle, and suppose both Q and E admit almost complex structures. Define the **orbifold Euler class** of E, $e_{\mathrm{orb}}(E)$, to be the image of the orbifold Euler class of VE under $\iota^* : H^*_{\mathrm{orb}}(E) \to H^*_{\mathrm{orb}}(Q)$,

$$e_{\mathrm{orb}}(E) := \iota^* e_{\mathrm{orb}}(VE).$$

By virtue of Lemma 9, this definition extends the original definition of $e_{\mathrm{orb}}(E)$ in [5].

We now are prepared to state and prove the main result of this section, that the orbifold Euler class acts as an obstruction to the existence of nonvanishing sections of vector bundles over orbifolds.

Theorem 12. *Let $\rho : E \to Q$ be an orbifold vector bundle over the orbifold Q. Suppose E admits a nonvanishing section s. Then $e_{\mathrm{orb}}(E) = 0 \in H^*_{\mathrm{orb}}(Q)$.*

Proof. Suppose $\tilde{E}$ is a good bundle over $\tilde{Q}$. Let S denote the trivial rank-1 subbundle of E given by the linear span of the image of s, and let $\tilde{S}$ denote the span of the nonvanishing section $\tilde{s}$ of $\tilde{E}$ induced by s. Endow E with a metric, and then E splits into $S \oplus S^{\perp}$. Choose a metric connection ∇ for E that respects this product structure, and then the induced connection $\tilde{\nabla}$ on $\tilde{E}$ respects the obvious splitting of $\tilde{E}$ into $\tilde{S} \oplus \tilde{S}^{\perp}$ using the induced metric on $\tilde{E}$.

Let ω denote the connection form for ∇, $\tilde{\omega}$ that for $\tilde{\nabla}$, and Ω, $\tilde{\Omega}$ the respective curvature forms (and $\tilde{\Omega}_{(g)}$ the curvature forms restricted to the sector corresponding to $(g) \in T$). Then with respect to a local orthonormal frame field $(e_1, \ldots, e_k)$ for E such that $e_1 \in S$, it is clear that $\omega_{i,j} = 0$ and $\Omega_{i,j} = 0$ whenever $i = 1$ or $j = 1$. Hence, $\tilde{\Omega}_{i,j} = 0$ whenever $i = 1$ or $j = 1$.

For each $(g) \in T$, the Euler curvature form that represents $e(\tilde{E}_{(g)}) \in H^*_{dR}(\tilde{Q}_{(g)})$ is defined by

$$E(\tilde{\Omega}_{(g)}) := \begin{cases} \dfrac{1}{2^{2m}\pi^m m!} \displaystyle\sum_{\tau \in S(l)} (-1)^{\tau} (\tilde{\Omega}_{(g)})_{\tau(1)\tau(2)} \wedge \cdots \wedge (\tilde{\Omega}_{(g)})_{\tau(l-1)\tau(l)}, & \text{if } l \text{ is even,} \\[2mm] 0, & \text{if } l \text{ is odd,} \end{cases}$$

where l is the rank of $\tilde{E}_{(g)}$ (which may be less than the rank k of E if $(g) \neq (1)$). In this case, as at least one of the $\tilde{\Omega}_{i,j}$ factors vanishes in each term, we have that $E(\tilde{\Omega}_{(g)}) = 0$. As this is true for each $(g) \in T$, $e_{\mathrm{orb}}(E) = 0$. Of course, given Lemma 9, this implies that $\iota^* e_{\mathrm{orb}}(VE) = 0$ as well.

If E is a bad bundle, then by Lemma 10, VE admits a nonvanishing section. By Lemma 4, VE is a good bundle, so the above argument implies that $e_{\mathrm{orb}}(VE) = 0 \in H^*_{\mathrm{orb}}(E)$. Hence $e_{\mathrm{orb}}(E) = 0 \in H^*_{\mathrm{orb}}(Q)$. $\quad \square$

We conclude this paper with an example illustrating Theorem 12 applied to a bundle whose ordinary Euler class $e(E)$ is trivial yet admits only the zero section.

Example 13. Consider the unreduced orbifold Q given by a smooth manifold diffeomorphic to S^2 equipped with a trivial $\mathbb{Z}/3\mathbb{Z}$-action. Since every point $p \in Q$ is fixed by a nontrivial group element, each point of the orbifold is singular. Now, consider the orbifold vector bundle E given by equipping the trivial real rank-2 bundle over S^2 with a fiber-wise $\mathbb{Z}/3\mathbb{Z}$-action using the standard representation of $\mathbb{Z}/3\mathbb{Z}$ on $\mathbb{R}^2$ as rotations. For the associated orbifold vector bundle E, each fiber is the quotient $\mathbb{R}^2/(\mathbb{Z}/3\mathbb{Z})$, in which the only fixed point is the zero vector. Hence, the only section of this bundle is the zero section.

Now, consider the vertical tangent space VE. It is easy to see that the orbifold Euler class $e_{\mathrm{orb}}(VE)$ of VE as a bundle over E has nonzero components in degree $H^{\frac{2}{3}}_{\mathrm{orb}}(E)$ and $H^{\frac{4}{3}}_{\mathrm{orb}}(E)$ corresponding to the twisted sectors $\tilde{E}_{(1)}$ and $\tilde{E}_{(2)}$ (which are both diffeomorphic to Q, and form trivial rank-0 bundles over $\tilde{Q}_{(1)}$ and $\tilde{Q}_{(2)}$, respectively).

In this case, the isomorphism $\iota^* : H^*_{\mathrm{orb}}(E) \to H^*_{\mathrm{orb}}(Q)$ maps $H^{\frac{2}{3}}_{\mathrm{orb}}(E)$ and $H^{\frac{4}{3}}_{\mathrm{orb}}(E)$ into $H^0_{\mathrm{orb}}(Q)$; the degree shifting number is $\frac{1}{3}$ for the generator $1 \in \mathbb{Z}/3\mathbb{Z}$ with respect to the orbifold structure of E, but is 0 with respect to the orbifold structure of Q. Hence, the orbifold Euler class $e_{\mathrm{orb}}(E) := \iota^* e_{\mathrm{orb}}(VE)$ of E in $H^*_{\mathrm{orb}}(Q)$ has nontrivial terms in degree 0.

Acknowledgements

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